

STUDENT'S t -TEST WITHOUT SYMMETRY CONDITIONS

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ABSTRACT. An explicit representation of an arbitrary zero-mean distribution as the mixture of (at-most-)two-point zero-mean distributions is given. Based in this representation, tests for (i) asymmetry patterns and (ii) for location without symmetry conditions can be constructed. Exact inequalities implying conservative properties of such tests are presented. These developments extend results established earlier by Efron, Eaton, and Pinelis under a symmetry condition.

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1. INTRODUCTION

Efron [3] considered the so-called self-normalized sum

$$(1.1) \quad S := \frac{X_1 + \cdots + X_n}{\sqrt{X_1^2 + \cdots + X_n^2}},$$

assuming that the X_i 's are any random variables (r.v.'s) satisfying the orthant symmetry condition: the joint distribution of $\eta_1 X_1, \dots, \eta_n X_n$ is the same for any choice of signs η_1, \dots, η_n in the set $\{1, -1\}$, so that, in particular, each X_i is symmetric(ally distributed). It suffices that the X_i 's be independent and symmetrically (but not necessarily identically) distributed. On the event $\{X_1 = \cdots = X_n = 0\}$, $S := 0$.

Following Efron [3], note that the conditional distribution of any symmetric r.v. X given $|X|$ is the symmetric distribution on the (at-most-)two-point set $\{|X|, -|X|\}$. Therefore, under the orthant symmetry condition, the distribution of S is the mixture of the distributions of the normalized Khinchin-Rademacher

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sums $\varepsilon_1 a_1 + \dots + \varepsilon_n a_n$, where the ε_i 's are independent Rademacher r.v.'s, with $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$ for all i , which are also independent of the X_i 's, and $a_i = X_i / (X_1^2 + \dots + X_n^2)^{\frac{1}{2}}$, so that $a_1^2 + \dots + a_n^2 = 1$ (except on the event $\{X_1 = \dots = X_n = 0\}$, where $a_1 = \dots = a_n = 0$).

Let $Z \sim N(0, 1)$. Let a_1, \dots, a_n be any real numbers such that $a_1^2 + \dots + a_n^2 = 1$. The sharp form,

$$(1.2) \quad \mathbb{E}f(\varepsilon_1 a_1 + \dots + \varepsilon_n a_n) \leq \mathbb{E}f(Z),$$

of Khinchin's inequality [7] for $f(x) \equiv |x|^p$ was proved by Whittle (1960) [14] for $p \geq 3$ and Haagerup (1982) [4] for $p \geq 2$. For $f(x) \equiv e^{\lambda x}$ ($\lambda \geq 0$), inequality (1.2) follows from Hoeffding (1963) [5], whence

$$(1.3) \quad P(\varepsilon_1 a_1 + \dots + \varepsilon_n a_n \geq x) \leq \inf_{\lambda \geq 0} \frac{\mathbb{E}e^{\lambda Z}}{e^{\lambda x}} = e^{-x^2/2} \quad \forall x \geq 0.$$

As noted by Efron, inequalities (1.2) and (1.3) together with the mentioned mixture representation imply

$$(1.4) \quad \mathbb{E}e^{\lambda S} \leq \mathbb{E}e^{\lambda Z} \quad \forall \lambda \geq 0$$

and

$$(1.5) \quad P(S \geq x) \leq e^{-x^2/2} \quad \forall x \geq 0.$$

These results can be easily restated in terms of Student's statistic T , which is a monotonic function of S , as noted by Efron: $T = \sqrt{\frac{n-1}{n}} S / \sqrt{1 - S^2/n}$.

Eaton (1970) [1] proved the Khinchin-Whittle-Haagerup inequality (1.2) for a much richer class of moment functions, which essentially coincides with the class \mathcal{F}^3 of all convex functions f with a convex second derivative f'' ; see [9, Proposition A.1] and also [12]. Based on this extension of (1.2), inequality (1.3) was improved in [1, 2, 9]. In particular, Pinelis (1994) [9] obtained the following improvement of a conjecture by Eaton (1974) [2]:

$$P(\varepsilon_1 a_1 + \dots + \varepsilon_n a_n \geq x) \leq \frac{2e^3}{9} P(Z \geq x) \quad \forall x \in \mathbb{R}.$$

Thus, inequalities (1.4) and (1.5) can be improved as follows:

$$(1.6) \quad \mathbb{E}f(S) \leq \mathbb{E}f(Z) \quad \forall f \in \mathcal{F}^3$$

and

$$(1.7) \quad P(S \geq x) \leq \frac{2e^3}{9} P(Z \geq x) \quad \forall x \in \mathbb{R}.$$

Multivariate extensions of these results, which can be expressed in terms of Hotelling's statistic in place of Student's, were also obtained in [9].

It was pointed out in [9, Theorem 2.8] that, since the normal tail decreases fast, inequality (1.7) implies that relevant quantiles of S may exceed the corresponding standard normal quantiles only by a relatively small amount, so that one can use (1.7) rather efficiently to test symmetry even for non-i.i.d. observations.

Here we shall present extensions of inequalities (1.6) and (1.7) to the case when the X_i 's are not symmetric. (Asymptotics for large deviations of S for i.i.d. X_i 's without moment conditions was obtained recently by Jing, Shao and Zhou [6].)

Our basic idea is to represent any zero-mean, possibly asymmetric distribution as an appropriate mixture of two-point zero-mean distributions. Let us assume at

first that a zero-mean r.v. X has an everywhere continuous and strictly increasing distribution function (d.f.). Consider the truncated r.v. $X_{a,b} := X \mathbf{I}\{a \leq X \leq b\}$. (Here and in what follows, as usual, $\mathbf{I}\{\mathcal{A}\}$ is the indicator of a given assertion \mathcal{A} , so that $\mathbf{I}\{\mathcal{A}\} = 1$ if \mathcal{A} is true and $\mathbf{I}\{\mathcal{A}\} = 0$ if \mathcal{A} is false.) Then, for every fixed $a \in (-\infty, 0]$, the function $b \mapsto \mathbf{E}X_{a,b}$ is continuous and increasing on the interval $[0, \infty)$ from $\mathbf{E}X_{a,0} \leq 0$ to $\mathbf{E}X_{a,\infty} > 0$. Hence, for each $a \in (-\infty, 0]$, there exists a unique value $b \in [0, \infty)$ such that $\mathbf{E}X_{a,b} = 0$. Similarly, for each $b \in [0, \infty)$, there exists a unique value $a \in (-\infty, 0]$ such that $\mathbf{E}X_{a,b} = 0$. That is, one has a one-to-one correspondence between $a \in (-\infty, 0]$ and $b \in [0, \infty)$ such that $\mathbf{E}X_{a,b} = 0$. Denote by $r := r_X$ the *reciprocating* function defined on \mathbb{R} and carrying this correspondence, so that

$$\mathbf{E}X \mathbf{I}\{X \text{ is between } x \text{ and } r(x)\} = 0 \quad \forall x \in \mathbb{R};$$

the function r is decreasing on \mathbb{R} and such that $r(r(x)) = x \quad \forall x \in \mathbb{R}$; moreover, $r(0) = 0$. (Clearly, $r(x) = -x$ for all real x if the r.v. X is symmetric.) Thus, the set $\{\{x, r(x)\} : x \in \mathbb{R}\}$ of (at-most-)two-point sets constitutes a partition of \mathbb{R} . Moreover, the two-point set $\{x, r(x)\}$ is uniquely determined by the distance $|x - r(x)| = |x| + |r(x)|$ between the two points, as well as by the product $|x| |r(x)|$. One can see that the conditional distribution of the zero-mean r.v. X given $W := |X - r(X)|$ (or, equivalently, $Y := |X| |r(X)|$) is the uniquely determined zero-mean distribution on the two-point set $\{X, r(X)\}$. Thus, the distribution of the zero-mean r.v. X with an everywhere positive density is represented as a mixture of two-point zero-mean distributions. This mixture is given rather explicitly, provided that the distribution of r.v. X is known.

Thus, one has generalized versions of the self-normalized sum (1.1), which require – instead of the symmetry of independent r.v.’s X_i – only that the X_i ’s be zero-mean:

$$S_W := \frac{X_1 + \cdots + X_n}{\frac{1}{2}\sqrt{W_1^2 + \cdots + W_n^2}} \quad \text{and} \quad S_{Y,\lambda} := \frac{X_1 + \cdots + X_n}{(Y_1^\lambda + \cdots + Y_n^\lambda)^{\frac{1}{2\lambda}}},$$

where $\lambda > 0$,

$$W_i := |X_i - r_i(X_i)| \quad \text{and} \quad Y_i := |X_i r_i(X_i)|,$$

and the reciprocating function $r_i := r_{X_i}$ is constructed as above, based on the distribution of X_i , for each i , so that the reciprocating functions r_i may be different from one another if the X_i ’s are not identically distributed. On the event $\{X_1 = \cdots = X_n = 0\}$ (which is the same as either one of events $\{W_1 = \cdots = W_n = 0\}$ and $\{Y_1 = \cdots = Y_n = 0\}$), $S_W := 0$ and $S_{Y,\lambda} := 0$. Note that $S_W = S_{Y,1} = S$ when the X_i ’s are symmetric. Logan *et al* [8] and Shao [13] obtained limit theorems for the “symmetric” version of $S_{Y,\lambda}$ (with the reciprocating function $r(x) \equiv -x$), whereas the X_i ’s did not need to be symmetric.

These constructions can be extended to the general case of any zero-mean r.v. X , possibly with a d.f. which is not continuous or strictly increasing. Toward that end, one can use randomization (by means of a r.v. uniformly distributed in interval $(0, 1)$) to deal with the atoms of the distribution of r.v. X , and generalized inverse functions to deal with the intervals on which the d.f. of X is constant.

Note that the reciprocating function r depends on the (usually unknown in statistics) distribution of the underlying r.v. X . However, if e.g. the X_i constitute an i.i.d. sample, then the function G defined by (2.1) can be estimated based on the sample, so that one can estimate the reciprocating function r . Thus, replacing

$X_1 + \dots + X_n$ in the numerators of S_W and $S_{Y,\lambda}$ by $X_1 + \dots + X_n - n\theta$, one obtains approximate pivots to be used to construct confidence intervals or, equivalently, tests for an unknown mean θ . One can also use bootstrap to estimate the distributions of such pivots.

2. RESULTS

Let X be a zero-mean real-valued r.v. defined on a probability space $(\Omega, \Sigma, \mathbf{P})$. Let

$$(2.1) \quad G(x) := \begin{cases} \mathbf{E}X \mathbf{I}\{X \in (0, x]\} & \text{if } x \in [0, \infty), \\ \mathbf{E}(-X) \mathbf{I}\{X \in [x, 0)\} & \text{if } x \in (-\infty, 0]. \end{cases}$$

Note that $G(0) = 0$; G is non-decreasing and right-continuous on $[0, \infty)$; and G is non-increasing and left-continuous on $(-\infty, 0]$; in particular, G is continuous at 0. Moreover, the condition $\mathbf{E}X = 0$ implies that

$$(2.2) \quad G(\infty) = G(-\infty) = \frac{1}{2}\mathbf{E}|X| =: m < \infty.$$

Thus, $G(x) \in [0, m]$ for all $x \in [-\infty, \infty]$.

For $h \in [0, m]$, let

$$(2.3) \quad x_+(h) := \inf\{x \in [0, \infty) : G(x) \geq h\},$$

$$(2.4) \quad x_-(h) := \sup\{x \in (-\infty, 0] : G(x) \geq h\}.$$

Note that $x_+(h) \in [0, \infty)$ and $x_-(h) \in (-\infty, 0]$ for all $h \in [0, m]$.

For $x \in \mathbb{R}$ and $u \in [0, 1]$, define the *reciprocating function* of r.v. X by the formula

$$(2.5) \quad r(x, u) := \begin{cases} x_-(H(x, u)) & \text{if } x \in [0, \infty), \\ x_+(H(x, u)) & \text{if } x \in (-\infty, 0], \end{cases}$$

where

$$(2.6) \quad H(x, u) := \begin{cases} G(x-) + u \cdot (G(x) - G(x-)) & \text{if } x \in [0, \infty), \\ G(x+) + u \cdot (G(x) - G(x+)) & \text{if } x \in (-\infty, 0]. \end{cases}$$

Note that $H(x, u)$ depends on u for a given value of x only if $\mathbf{P}(X = x) \neq 0$.

Let $U : \Omega \rightarrow \mathbb{R}$ be a r.v. uniformly distributed on the unit interval $[0, 1]$ and independent of X . For a real x , let

$$(2.7) \quad U_x := \begin{cases} U & \text{if } \mathbf{P}(X = x) \neq 0, \\ 1 & \text{if } \mathbf{P}(X = x) = 0. \end{cases}$$

Introduce the r.v.'s

$$(2.8) \quad W := |X - r(X, U_X)| \quad \text{and} \quad Y := |X r(X, U_X)|$$

where the r.v. U_X is defined in the usual manner: $U_X(\omega) := U_{X(\omega)}(\omega)$, for all $\omega \in \Omega$.

Theorem 2.1. (i): *There exist an event $\Omega_0 \in \Sigma$ such that $\mathbf{P}(\Omega_0) = 1$ and continuous functions $c : V_0 \rightarrow (-\infty, 0]$ and $d : V_0 \rightarrow [0, \infty)$ defined on the set $V_0 := \{W(\omega) : \omega \in \Omega_0\}$ such that d and $(-c)$ are nondecreasing on V_0 , and on Ω_0 one has*

$$\{X, r(X, U_X)\} = \{c(W), d(W)\} \quad \text{and} \quad d(W) - c(W) = W.$$

(ii): the conditional distribution of X given W coincides with that of D_W :

$$(2.9) \quad \mathcal{L}(X|W) = \mathcal{L}(D_W|W),$$

where, for every $v \in V_0$, D_v is a r.v. such that

$$D_v = \begin{cases} d(v) & \text{with probability } \frac{|c(v)|}{|c(v)|+d(v)}, \\ c(v) & \text{with probability } \frac{d(v)}{|c(v)|+d(v)} \end{cases}$$

if $v \neq 0$, and $D_0 \equiv 0$, so that D_v takes on at most two distinct values and

$$\mathbb{E}D_v = 0.$$

Formally, (2.9) is understood as follows:

$$(2.10) \quad \mathbb{E}f(X) \mathbf{I}\{W \in B\} = \mathbb{E}\varphi_f(W) \mathbf{I}\{W \in B\}$$

for all Borel functions $f: \mathbb{R} \rightarrow [0, \infty)$ and all Borel sets $B \subseteq [0, \infty)$, where

$$(2.11) \quad \varphi_f(v) := \mathbb{E}f(D_v) = \begin{cases} f(c(v)) \frac{d(v)}{|c(v)|+d(v)} + f(d(v)) \frac{|c(v)|}{|c(v)|+d(v)} & \text{if } v \neq 0, \\ f(0) & \text{if } v = 0. \end{cases}$$

That is, (2.9) means that

$$(2.12) \quad \mathbb{E}f(X) \mathbf{I}\{W \in B\} = \int_{\mathbb{R}} \mathbb{P}(W \in dv) \mathbb{E}f(D_v) \mathbf{I}\{v \in B\},$$

where f and B are as in (2.10).

This understanding differs somewhat from the way in which the notion of the conditional distribution is usually understood. The above meaning is more convenient in the applications below, because (2.11) can be generalized as follows.

For all Borel functions $F: \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$,

$$(2.13) \quad \mathbb{E}F(X, W) = \int_{\mathbb{R}} \mathbb{P}(W \in dv) \mathbb{E}F(D_v, v);$$

in fact, one can write $\int_{[0, \infty)}$ instead of $\int_{\mathbb{R}}$ in (2.12) and (2.13), because $W \geq 0$ a.s.

The following theorem is quite similar to Theorem 2.1.

Theorem 2.2. (i): There exist an event $\Omega_0 \in \Sigma$ such that $\mathbb{P}(\Omega_0) = 1$ and continuous functions $\tilde{c}: \tilde{V}_0 \rightarrow (-\infty, 0]$ and $\tilde{d}: \tilde{V}_0 \rightarrow [0, \infty)$ defined on the set $\tilde{V}_0 := \{Y(\omega): \omega \in \Omega_0\}$ such that \tilde{d} and $(-\tilde{c})$ are nondecreasing on \tilde{V}_0 , and on Ω_0 one has

$$\{X, r(X, U_X)\} = \{\tilde{c}(Y), \tilde{d}(Y)\} \quad \text{and} \quad -\tilde{c}(Y)\tilde{d}(Y) = Y.$$

(ii): the conditional distribution of X given Y coincides with that of \tilde{D}_Y :

$$(2.14) \quad \mathcal{L}(X|Y) = \mathcal{L}(\tilde{D}_Y|Y),$$

where, for every $y \in \tilde{V}_0$, \tilde{D}_y is a r.v. such that

$$\tilde{D}_y = \begin{cases} \tilde{d}(y) & \text{with probability } \frac{|\tilde{c}(y)|}{|\tilde{c}(y)|+\tilde{d}(y)}, \\ \tilde{c}(y) & \text{with probability } \frac{\tilde{d}(y)}{|\tilde{c}(y)|+\tilde{d}(y)} \end{cases}$$

if $y \neq 0$, and $\tilde{D}_0 \equiv 0$, so that \tilde{D}_y takes on at most two distinct values and

$$\mathbb{E}\tilde{D}_y = 0.$$

Formally, (2.14) is understood as follows.

For all Borel functions $F: \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$,

$$(2.15) \quad \mathbb{E}F(X, Y) = \int_{\mathbb{R}} \mathbb{P}(Y \in dy) \mathbb{E}F(\tilde{D}_y, y);$$

in fact, one can write $\int_{[0, \infty)}$ instead of $\int_{\mathbb{R}}$ in (2.15), because $Y \geq 0$ a.s.

Remark. It is easily seen from the proof of Theorem 2.2 or, more specifically, from the proof of Lemma 3.9, that Theorem 2.2 holds for all r.v.'s Y of the more general form $\psi(|X|, |r(X, U_X)|)$, where $\psi(u, v)$ is any expression such that (i) $\psi(0, 0) = 0$; (ii) $\psi(u, v)$ is nondecreasing in u and in v over all nonnegative u and v ; and (iii) $\psi(u, v)$ is strictly increasing in u and in v over all strictly positive u and v .

Example 1. Let X have the discrete distribution $\frac{5}{10} \delta_{-1} + \frac{1}{10} \delta_0 + \frac{3}{10} \delta_1 + \frac{1}{10} \delta_2$ on the finite set $\{-1, 0, 1, 2\}$, where δ_a denotes the (Dirac) probability distribution on the singleton set $\{a\}$. Then $m = \frac{5}{10}$ and, for $x \in \mathbb{R}$, $u \in (0, 1)$, and $h \in [0, m]$,

$$\begin{aligned} G(x) &= \frac{5}{10} \mathbf{I}\{x \leq -1\} + \frac{3}{10} \mathbf{I}\{1 \leq x < 2\} + \frac{5}{10} \mathbf{I}\{2 \leq x\}, \\ x_+(h) &= \mathbf{I}\{0 < h \leq \frac{3}{10}\} + 2 \mathbf{I}\{\frac{3}{10} < h\}, \quad x_-(h) = -\mathbf{I}\{0 < h\}, \\ H(-1, u) &= \frac{5}{10} u, \quad H(0, u) = 0, \quad H(1, u) = \frac{3}{10} u, \quad H(2, u) = \frac{3}{10} + \frac{2}{10} u, \\ r(-1, u) &= \mathbf{I}\{u \leq \frac{3}{5}\} + 2 \mathbf{I}\{u > \frac{3}{5}\}, \quad r(0, u) = 0, \quad r(1, u) = -1, \quad r(2, u) = -1. \end{aligned}$$

Therefore, the distribution of W is $\frac{1}{10} \delta_0 + \frac{6}{10} \delta_2 + \frac{3}{10} \delta_3$ and the conditional distributions of X given $W = 0$, $W = 2$, and $W = 3$ are δ_0 , $\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1$, and $\frac{2}{3} \delta_{-1} + \frac{1}{3} \delta_2$, respectively. Thus, the zero-mean distribution of X is represented as a mixture of (at-most-)two-point zero-mean distributions:

$$\frac{5}{10} \delta_{-1} + \frac{1}{10} \delta_0 + \frac{3}{10} \delta_1 + \frac{1}{10} \delta_2 = \frac{1}{10} \delta_0 + \frac{6}{10} (\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1) + \frac{3}{10} (\frac{2}{3} \delta_{-1} + \frac{1}{3} \delta_2).$$

Equivalently, one can condition here on Y instead of W . The distribution of Y is $\frac{1}{10} \delta_0 + \frac{6}{10} \delta_1 + \frac{3}{10} \delta_2$ and the conditional distributions of X given $Y = 0$, $Y = 1$, and $Y = 2$ are δ_0 , $\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1$, and $\frac{2}{3} \delta_{-1} + \frac{1}{3} \delta_2$, respectively.

Remark. A zero-mean distribution can be represented as a mixture of (at-most-)two-point zero-mean distributions in a variety of ways. For instance, the symmetric distribution $\frac{1}{10} \delta_{-2} + \frac{4}{10} \delta_{-1} + \frac{4}{10} \delta_1 + \frac{1}{10} \delta_2$ can be represented either as the mixture $\frac{3}{10} (\frac{1}{3} \delta_{-2} + \frac{2}{3} \delta_1) + \frac{3}{10} (\frac{1}{3} \delta_2 + \frac{2}{3} \delta_{-1}) + \frac{4}{10} (\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1)$ of two asymmetric and one symmetric two-point zero-mean distributions or as the mixture $\frac{1}{5} (\frac{1}{2} \delta_{-2} + \frac{1}{2} \delta_2) + \frac{4}{5} (\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1)$ of two symmetric two-point zero-mean distributions. The latter, “more symmetric” representation coincides with the one produced by the method of Theorem 2.1 (or, equivalently, by that of Theorem 2.2). It appears that in general this method will produce the mixture representation that is “the most symmetric” in an appropriate sense, and hence the best with respect to such applications as Corollaries 2.4 and 2.6, given below.

Let us now apply Theorems 2.1 and 2.2 to the mentioned asymmetry-corrected versions of self-normalized sums.

Theorem 2.3. Suppose that X_1, \dots, X_n are independent zero-mean r.v.'s and U_1, \dots, U_n are independent r.v.'s uniformly distributed on $[0, 1]$, which are also independent of X_1, \dots, X_n . For each $i = 1, \dots, n$, let

$$W_i := |X_i - r_i(X_i, (U_i)_{X_i})|$$

be a r.v. constructed based on X_i and U_i the way the r.v. $W = |X - \mathbf{r}(X, U_X)|$ was constructed in (2.7) and (2.8) based on X and U , where \mathbf{r}_i is the reciprocating function for (the distribution of) r.v. X_i . Let

$$S_W := \frac{X_1 + \cdots + X_n}{\frac{1}{2}\sqrt{W_1^2 + \cdots + W_n^2}},$$

where the rule $\frac{0}{0} := 0$ is used if the denominator is zero. Then for every nonnegative Borel function f on \mathbb{R}

$$(2.16) \quad \mathbf{E}f(S_W) \leq \max(f(0), \sup \mathbf{E}f(Z_1 + \cdots + Z_n)),$$

where the sup is taken over all n -tuples of independent zero-mean r.v.'s Z_1, \dots, Z_n with the property that each Z_i takes on only two values, say c_i and d_i , such that

$$\frac{1}{2}\sqrt{(d_1 - c_1)^2 + \cdots + (d_n - c_n)^2} = 1.$$

For every natural α , let \mathcal{H}_+^α denote the class of all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that f has finite derivatives $f^{(0)} := f, f^{(1)} := f', \dots, f^{(\alpha-1)}$ on \mathbb{R} , $f^{(\alpha-1)}$ is convex on \mathbb{R} , and $f^{(j)}(-\infty+) = 0$ for $j = 0, 1, \dots, \alpha - 1$.

Corollary 2.4. *Under the conditions of Theorem 2.3,*

$$\begin{aligned} \mathbf{E}f(S_W) &\leq \mathbf{E}f(Z) \quad \forall f \in \mathcal{H}_+^5 \quad \text{and} \\ \mathbf{P}(S_W \geq x) &\leq c_{5,0}\mathbf{P}(Z \geq x) \quad \forall x \in \mathbb{R}, \end{aligned}$$

where $c_{5,0} = 5!(e/5)^5 = 5.699\dots$

This follows immediately from Theorem 2.3 and results of [11]. (Note that every function $f \in \mathcal{H}_+^5$ is convex, and so, by Jensen's inequality, $f(0) \leq \mathbf{E}f(Z)$.)

The following theorem is quite similar to Theorem 2.3.

Theorem 2.5. *With the X_i 's and U_i 's as in Theorem 2.3, let for each $i = 1, \dots, n$*

$$Y_i := |X_i \mathbf{r}_i(X_i, (U_i)_{X_i})|,$$

where \mathbf{r}_i is the reciprocating function for r.v. X_i . For any $\lambda > 0$, let

$$S_{Y,\lambda} := \frac{X_1 + \cdots + X_n}{(Y_1^\lambda + \cdots + Y_n^\lambda)^{\frac{1}{2\lambda}}},$$

where the rule $\frac{0}{0} := 0$ is used if the denominator is zero. Then for every nonnegative Borel function f on \mathbb{R}

$$\mathbf{E}f(S_{Y,\lambda}) \leq \max(f(0), \sup \mathbf{E}f(Z_1 + \cdots + Z_n)),$$

where the sup is taken over all n -tuples of independent zero-mean r.v.'s Z_1, \dots, Z_n with the property that each Z_i takes on only two values, say c_i and d_i , such that

$$|c_1 d_1|^\lambda + \cdots + |c_n d_n|^\lambda = 1.$$

(Note that $\text{Var}Z_i = |c_i d_i|$ for all i .)

Corollary 2.6. *Under the conditions of Theorem 2.5, suppose that for some $p \in (0, 1)$ and all $i \in \{1, \dots, n\}$*

$$(2.17) \quad \frac{X_i}{|\mathbf{r}_i(X_i, (U_i)_{X_i})|} \mathbf{I}\{X_i > 0\} \leq \frac{1-p}{p} \quad \text{a.s.}$$

Then for all

$$(2.18) \quad \lambda \geq \lambda_*(p) := \begin{cases} \frac{1+p+2p^2}{2(\sqrt{p-p^2}+2p^2)} & \text{if } 0 < p \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq p < 1. \end{cases}$$

one has

$$\begin{aligned} \mathbb{E}f(V_{Y,\lambda}) &\leq \mathbb{E}f(T_n) \quad \forall f \in \mathcal{H}_+^3 \quad \text{and} \\ \mathbb{P}(V_{Y,\lambda} \geq x) &\leq c_{3,0} \mathbb{P}^{\text{LC}}(T_n \geq x) \quad \forall x \in \mathbb{R}, \end{aligned}$$

where $T_n := (Z_1 + \dots + Z_n)/n^{1/(2\lambda)}$; Z_1, \dots, Z_n are independent r.v.'s each having the standardized Bernoulli distribution with parameter p ; the function $x \mapsto \mathbb{P}^{\text{LC}}(T_n \geq x)$ is the least log-concave majorant of the function $x \mapsto \mathbb{P}(T_n \geq x)$ on \mathbb{R} ; $c_{3,0} = 2e^3/9 = 4.4634\dots$. The upper bound $c_{3,0} \mathbb{P}^{\text{LC}}(T_n \geq x)$ can be replaced by somewhat better ones, in accordance with [10, Theorem 2.3] or [12, (3.3)]. The lower bound $\lambda_*(p)$ on λ given by (2.18) is the best possible one, for each p .

Condition (2.17) is likely to hold when the X_i 's are bounded i.i.d. r.v.'s. For instance, (2.17) holds with $p = \frac{1}{3}$ for r.v. X in Example 1 in place of X_i .

Corollary 2.6 follows immediately from Theorem 2.3 and results of [12].

3. PROOFS

We shall precede the proof of the theorems by the statements of a number of lemmas (in Subsection 3.1). Next, we shall prove the theorems (in Subsection 3.2). Finally, we shall prove the lemmas (in Subsection 3.3).

3.1. Statements of lemmata. Without loss of generality, one may assume that in Theorem 2.1

$$\mathbb{P}(X = 0) \neq 1.$$

Hence,

$$m \in (0, \infty).$$

To state our lemmas, we need to introduce more notation. Consider the sets

$$\begin{aligned} M_+ &:= \{x \in (0, \infty) : \forall y < x \, \mathbb{P}(X \in (y, x]) > 0\}, \\ N_+ &:= \{x \in (0, \infty) : \mathbb{P}(X = x) = 0\}, \\ L_+ &:= \{x \in (0, \infty) : \exists y < x \, \mathbb{P}(X \in (y, x)) = 0\}, \\ &= \{x \in (0, \infty) : \exists y \in [0, x) \, \mathbb{P}(X \in (y, x)) = 0\}, \\ M_- &:= \{x \in (-\infty, 0) : \forall y > x \, \mathbb{P}(X \in [x, y)) > 0\}, \\ N_- &:= \{x \in (-\infty, 0) : \mathbb{P}(X = x) = 0\}, \\ L_- &:= \{x \in (-\infty, 0) : \exists y > x \, \mathbb{P}(X \in (x, y)) = 0\} \\ &= \{x \in (-\infty, 0) : \exists y \in (x, 0] \, \mathbb{P}(X \in (x, y)) = 0\}, \\ M &:= M_+ \cup M_-, \\ N &:= N_+ \cup N_-, \\ L &:= L_+ \cup L_-. \end{aligned}$$

Note that

$$(3.1.1) \quad N_+ \cap L_+ = (0, \infty) \setminus M_+, \quad N_- \cap L_- = (-\infty, 0) \setminus M_-.$$

Now we can introduce the sets

$$\begin{aligned} \mathcal{G}_+ &:= \left\{ (x, u) : \right. \\ (3.1.2a) \quad & \quad \quad x \in M_+, \ 0 \leq u \leq 1, \\ (3.1.2b) \quad & \quad \quad x \in N_+ \implies u = 1, \\ (3.1.2c) \quad & \quad \quad x \in L_+ \implies u > 0, \\ (3.1.2d) \quad & \quad \quad \left. \mathbb{P}(X > x) = 0 \implies (x \notin N_+ \ \& \ u < 1) \right\}, \end{aligned}$$

$$\begin{aligned} \mathcal{G}_- &:= \left\{ (x, u) : \right. \\ (3.1.3a) \quad & \quad \quad x \in M_-, \ 0 \leq u \leq 1, \\ (3.1.3b) \quad & \quad \quad x \in N_- \implies u = 1, \\ (3.1.3c) \quad & \quad \quad x \in L_- \implies u > 0, \\ (3.1.3d) \quad & \quad \quad \left. \mathbb{P}(X < x) = 0 \implies (x \notin N_- \ \& \ u < 1) \right\}, \end{aligned}$$

$$(3.1.4) \quad \mathcal{G} := \mathcal{G}_+ \cup \mathcal{G}_-.$$

Note that

$$(3.1.5) \quad \mathcal{G}_+ \cap \mathcal{G}_- = \emptyset,$$

because $M_+ \cap M_- \subseteq (0, \infty) \cap (-\infty, 0) = \emptyset$ and in view of (3.1.2a) and (3.1.3a).

Lemma 3.1.

$$\mathbb{P}(X \notin M \cup \{0\}) = 0.$$

Lemma 3.2. (Recall definitions (2.2), (2.3), and (2.4).) For $h \in (0, m]$

$$(3.1.6) \quad x_+(h) = \min\{x \in (0, \infty] : G(x) \geq h\};$$

$$(3.1.7) \quad x_-(h) = \max\{x \in [-\infty, 0) : G(x) \geq h\};$$

$$(3.1.8) \quad G(y) < h \ \forall y \in [0, x_+(h)), \quad G(x_+(h)) \geq h \geq G(x_+(h)-);$$

$$(3.1.9) \quad G(y) < h \ \forall y \in (x_-(h), 0], \quad G(x_-(h)) \geq h \geq G(x_-(h)+).$$

If, moreover, $h \in (0, m)$ then

$$x_+(h) \in (0, \infty) \quad \text{and} \quad x_-(h) \in (-\infty, 0).$$

For

$$h \in (0, m],$$

let

$$(3.1.10) \quad \begin{aligned} u_+(h) &:= \begin{cases} \frac{h - G(x_+(h)-)}{G(x_+(h)) - G(x_+(h)-)} & \text{if } x_+(h) \notin N_+, \\ 1 & \text{otherwise,} \end{cases} \\ u_-(h) &:= \begin{cases} \frac{h - G(x_-(h)+)}{G(x_-(h)) - G(x_-(h)+)} & \text{if } x_-(h) \notin N_+, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 3.3. *The formula*

$$(3.1.11) \quad (0, m) \ni h \mapsto (x_+(h), u_+(h)) \in \mathcal{G}_+$$

defines a one-to-one map of the interval $(0, m)$ onto \mathcal{G}_+ , and the inverse map is given by the formula

$$(3.1.12) \quad \mathcal{G}_+ \ni (x, u) \mapsto h_+(x, u) := G(x-) + u \cdot (G(x) - G(x-)) \in (0, m).$$

Similarly, the formula

$$(3.1.13) \quad (0, m) \ni h \mapsto (x_-(h), u_-(h)) \in \mathcal{G}_-$$

defines a one-to-one map of the interval $(0, m)$ onto \mathcal{G}_- , and the inverse map is given by the formula

$$(3.1.14) \quad \mathcal{G}_- \ni (x, u) \mapsto h_-(x, u) := G(x+) + u \cdot (G(x) - G(x+)) \in (0, m).$$

Note that

$$H(x, u) = \begin{cases} h_+(x, u) & \text{for } (x, u) \in \mathcal{G}_+, \\ h_-(x, u) & \text{for } (x, u) \in \mathcal{G}_-, \end{cases}$$

where $H(x, u)$ is given by (2.6).

Now, using maps (3.1.11) and (3.1.13) and their inverses (3.1.12) and (3.1.14), one can define a one-to-one map of \mathcal{G} onto \mathcal{G}

$$(3.1.15) \quad \mathcal{G} \ni (x, u) \longleftrightarrow (\hat{x}, \hat{u}) \in \mathcal{G}$$

by formulas

$$(3.1.16) \quad (\hat{x}, \hat{u}) := \begin{cases} (x_-(h_+(x, u)), u_-(h_+(x, u))) & \text{if } (x, u) \in \mathcal{G}_+, \\ (x_+(h_-(x, u)), u_+(h_-(x, u))) & \text{if } (x, u) \in \mathcal{G}_-. \end{cases}$$

Thus, the one-to-one map (3.1.15) is inverse to itself. It maps \mathcal{G}_+ onto \mathcal{G}_- and \mathcal{G}_- onto \mathcal{G}_+ , and the latter two correspondences can be presented as follows:

$$\begin{aligned} \mathcal{G}_+ \ni (x, u) &\longleftrightarrow h = h_+(x, u) = h_-(\hat{x}, \hat{u}) \longleftrightarrow (\hat{x}, \hat{u}) \in \mathcal{G}_-, \\ \mathcal{G}_- \ni (x, u) &\longleftrightarrow h = h_-(x, u) = h_+(\hat{x}, \hat{u}) \longleftrightarrow (\hat{x}, \hat{u}) \in \mathcal{G}_+. \end{aligned}$$

Remark 3.4. For \hat{x} defined by (3.1.16) and r defined by (2.5), one has

$$r(x, u) = \hat{x}$$

for all $(x, u) \in \mathcal{G}$.

Let us now introduce the map

$$(3.1.17) \quad [0, m] \ni h \mapsto w(h) := x_+(h) - x_-(h).$$

Introduce also the set

$$(3.1.18) \quad V := \{w(h) : h \in (0, m)\}.$$

Lemma 3.5. *The functions x_+ , $(-x_-)$, and w are nonnegative and nondecreasing on $[0, m]$, and positive and left-continuous on $(0, m]$.*

Lemma 3.6. *Assume that $w(h_2) = w(h_1) + \varepsilon$ for some $\varepsilon \in [0, \infty)$ and some h_1 and h_2 in $[0, m]$. Then*

$$0 \leq x_+(h_2) - x_+(h_1) \leq \varepsilon, \quad 0 \leq x_-(h_1) - x_-(h_2) \leq \varepsilon.$$

As an immediate corollary to Lemma 3.6, one obtains the following.

Lemma 3.7. *If $w(h_2) = w(h_1)$ for some h_1, h_2 in $[0, m]$, then $x_+(h_2) = x_+(h_1)$ and $x_-(h_2) = x_-(h_1)$. Thus, for $h \in [0, m]$, the values of $x_+(h)$ and $x_-(h)$ are uniquely determined by the value of $w(h)$. Moreover, there are nonnegative nondecreasing continuous real functions, say $-c$ and d , defined on $V \cup \{0\}$ (see (3.1.18)) such that for all $h \in [0, m]$*

$$(3.1.19) \quad x_+(h) = d(w(h)) \quad x_-(h) = c(w(h)), \quad \text{and} \quad d(w(h)) - c(w(h)) = w(h).$$

Furthermore, by Lemma 3.5, the functions c and d vanish only at 0 and are Lipschitz with Lipschitz constants ≤ 1 .

Remark 3.8. Take any pair $(x, u) \in \mathcal{G}$. It follows from Lemma 3.3, Lemma 3.7, Remark 3.4, (3.1.17), and (3.1.18) that $v := |x - r(x, u)| \in V$. Moreover, $v = d(v) - c(v)$ and

- if $x > 0$, then $x = d(v)$ and $r(x, u) = c(v)$;
- if $x < 0$, then $x = c(v)$ and $r(x, u) = d(v)$.

Lemma 3.9. *There is a strictly increasing function $\tau: V \cup \{0\} \rightarrow \mathbb{R}$ such that*

$$(3.1.20) \quad |x| |r(x, u)| = \tau(|x - r(x, u)|) \text{ for all } (x, u) \in \mathcal{G}, \text{ and } \tau(0) = 0.$$

For $v \in V$, let

$$h_v := \sup\{h \in (0, m]: w(h) \leq v\}.$$

By the definition (3.1.18) of V , the set $\{h \in (0, m]: w(h) \leq v\}$ is non-empty. Moreover, by Lemma 3.5, the function w is left-continuous and nondecreasing on $(0, m]$. Therefore

$$(3.1.21) \quad h_v = \max\{h \in (0, m]: w(h) \leq v\}$$

and

$$(3.1.22) \quad w(h_v) = v.$$

Lemma 3.10. *For any $v \in V$,*

(i): *if $(x, u) \in \mathcal{G}_+$, then*

$$|x - r(x, u)| \leq v \iff \left(x < d(v) \text{ or } (x = d(v) \ \& \ h_+(x, u) \leq h_v) \right);$$

(ii): *if $(x, u) \in \mathcal{G}_-$, then*

$$|x - r(x, u)| \leq v \iff \left(x > c(v) \text{ or } (x = c(v) \ \& \ h_-(x, u) \leq h_v) \right);$$

Remark. It can be seen from the proof of Lemma 3.10 (or otherwise) that the condition $\left(x < d(v) \text{ or } (x = d(v) \ \& \ h_+(x, u) \leq h_v) \right)$ can be replaced by the seemingly simpler one: $(x \leq d(v) \ \& \ h_+(x, u) \leq h_v)$. However, the form used in the formulation of Lemma 3.10 will be more convenient when Lemma 3.10 is applied. A similar comment can be made concerning the corresponding condition in part (ii) of Lemma 3.10.

Lemma 3.11. *Let X and U_X be as in Theorem 2.1. Then*

$$P(X \neq 0, (X, U_X) \notin \mathcal{G}) = 0.$$

Lemma 3.12. *Let X and U_X be as in Theorem 2.1. Then for all $v \in [0, \infty)$*

$$EX \mathbf{I}\{W \leq v\} = 0;$$

recall the definition (2.8) of W .

Lemma 3.13. *Let X and U_X be as in Theorem 2.1. Then Lemma 3.12 can be generalized as follows: for any Borel set $B \subseteq [0, \infty)$,*

$$\mathbf{E} X \mathbf{I}\{W \in B\} = 0.$$

Let us say that a Borel set $C \subset (0, \infty)$ is *null* if $\mathbf{P}(W \in C) = 0$. Note that, if B is a null set, then identity (2.10) holds, because both sides of it are zero.

In the case when a Borel set $C \subset (0, \infty)$ is not null, it must contain a point $v \in V$. (Indeed, by Remark 3.8, the range of W on the event $\{(X, U_X) \in \mathcal{G}\}$ is contained in V . Also, by Lemma 3.11, the event $\{X \neq 0, (X, U_X) \notin \mathcal{G}\}$ is of zero probability. Finally, $W \in C \subset (0, \infty)$ implies $W \neq 0$ and hence $X \neq 0$.)

In the case when a bounded Borel set $C \subset (0, \infty)$ is not null, let

$$\begin{aligned} d_{\max}(C) &:= \sup\{d(v) : v \in V \cap C\}, \\ d_{\min}(C) &:= \inf\{d(v) : v \in V \cap C\}, \\ c_{\max}(C) &:= \sup\{c(v) : v \in V \cap C\}, \\ c_{\min}(C) &:= \inf\{c(v) : v \in V \cap C\}; \end{aligned}$$

note that, by Lemma 3.7, the first two of these four numbers are in $[0, \infty)$, while the last two of them are in $(-\infty, 0]$.

In addition, for any Borel function $f : \mathbb{R} \rightarrow [0, \infty)$, let

$$\begin{aligned} f_{r,\max}(C) &:= \sup\{f(x) : x \in [d_{\min}(C), d_{\max}(C)]\}, \\ f_{r,\min}(C) &:= \inf\{f(x) : x \in [d_{\min}(C), d_{\max}(C)]\}, \\ f_{\ell,\max}(C) &:= \sup\{f(x) : x \in [c_{\min}(C), c_{\max}(C)]\}, \\ f_{\ell,\min}(C) &:= \inf\{f(x) : x \in [c_{\min}(C), c_{\max}(C)]\}. \end{aligned}$$

Here, r and ℓ stand for “right” and “left”, respectively.

For any $\varepsilon > 0$, let us say that a bounded Borel set $C \subset (0, \infty)$ is (d, ε) -*good* if it is not null and is such that

$$0 < d_{\max}(C) \leq e^\varepsilon d_{\min}(C).$$

Similarly, let us say that a bounded Borel set C is (c, ε) -*good* if it is not null and is such that

$$0 < -c_{\min}(C) \leq e^\varepsilon (-c_{\max}(C));$$

recall that $c_{\min}(C) \leq c_{\max}(C) \leq 0$, for any $C \subset (0, \infty)$.

Let us say that a bounded Borel set $C \subset (0, \infty)$ is (f, ε) -*good* if it is not null and is such that

$$0 < f_{r,\max}(C) \leq e^\varepsilon f_{r,\min}(C) \quad \text{and} \quad 0 < f_{\ell,\max}(C) \leq e^\varepsilon f_{\ell,\min}(C).$$

Let us say that C is ε -*good* if it is (d, ε) -good, (c, ε) -good, and (f, ε) -good.

Let us say that a partition of a bounded Borel set B is *Borel* if every member of the partition is a Borel set. Let us say that such a partition is (d, ε) -*good* if every member set of the partition is either null or (d, ε) -good. Similarly defined are (c, ε) -good, (f, ε) -good, and ε -good partitions.

Lemma 3.14. *For any bounded Borel set $B \subset (0, \infty)$, any $\varepsilon \in (0, \infty)$, and any everywhere strictly positive and continuous function f , there always exists an ε -good partition of B .*

Lemma 3.15. *For any Borel function $f: \mathbb{R} \rightarrow [0, \infty)$, any bounded Borel set $C \subset (0, \infty)$, and any $\varepsilon \in (0, \infty)$, if C is null or ε -good, then (recall (2.11))*

$$(3.1.23) \quad \begin{aligned} e^{-4\varepsilon} \mathbb{E}\varphi_f(W) \mathbf{I}\{W \in C\} &\leq \mathbb{E}f(X) \mathbf{I}\{W \in C\} \\ &\leq e^{4\varepsilon} \mathbb{E}\varphi_f(W) \mathbf{I}\{W \in C\}. \end{aligned}$$

Let

$$D_{v_1}^{(1)}, \dots, D_{v_n}^{(n)}$$

be independent r.v.'s such that, for each $j \in \{1, \dots, n\}$, the r.v. $D_{v_j}^{(j)}$ is constructed based on the distribution of X_j the way the r.v. D_v was constructed in Theorem 2.1 based on the distribution of X .

Lemma 3.16. *Let $F(x_1, v_1, \dots, x_n, v_n)$ be a nonnegative Borel function of its $2n$ real arguments. Let $X_1, \dots, X_n, W_1, \dots, W_n$ be as in Theorem 2.3. Then identity (2.13) can be generalized as follows:*

$$(3.1.24) \quad \begin{aligned} &\mathbb{E}F(X_1, W_1, \dots, X_n, W_n) \\ &= \int_{\mathbb{R}^n} \left(\prod_{i=1}^n \mathbb{P}(W_i \in dv_i) \right) \mathbb{E}F(D_{v_1}^{(1)}, v_1, \dots, D_{v_n}^{(n)}, v_n). \end{aligned}$$

3.2. Proofs of the theorems.

Proof of Theorem 2.1. (i) Let $\Omega_0 := \{X = 0\} \cup \{(X, U_X) \in \mathcal{G}\}$. Then, by Lemma 3.11, one has $\mathbb{P}(\Omega_0) = 1$ and, by Remark 3.8, $V_0 \subseteq V \cup \{0\}$. The rest of part (i) of Theorem 2.1 now follows by Remark 3.8 and Lemma 3.7.

(ii) Here we need to prove identities (2.10) and (2.13). We shall do this in a few steps.

Step 1. Here we shall prove (2.10) assuming that (a) the function f is continuous and strictly positive everywhere on \mathbb{R} and (b) the Borel set B is a bounded subset of $(0, \infty)$.

By Lemma 3.14, for any $\varepsilon \in (0, \infty)$, there exists an ε -good partition of B . Applying Lemma 3.15 to every member set of such a partition and then summing over all the member sets, one sees that inequalities (3.1.23) hold for the entire set B , in place of C .

Since $\varepsilon > 0$ was chosen arbitrarily, this implies that (2.10) holds whenever the function f is continuous and strictly positive everywhere on \mathbb{R} and B is a bounded Borel subset of $(0, \infty)$. Thus, Step 1 of the proof of (2.10) is now complete.

Step 2. If B is any Borel subset of $(0, \infty)$, then the sets $B_n := B \cap (0, n]$ are bounded for all $n \in (0, \infty)$, so that, according to Step 1, (2.10) holds with B_n in place of B . It remains to let $n \rightarrow \infty$ to see that (2.10) holds whenever the function f is continuous and strictly positive everywhere on \mathbb{R} and the set B is any Borel subset of $(0, \infty)$.

Step 3. By (2.5), if $x \neq 0$, then $-r(x, u)$ is either 0 or of the same sign as x . Hence, one always has $|W| = |X - r(X, U_X)| \geq |X|$, so that $W = 0$ always implies $X = 0$. Therefore and in view of (2.11), identity (2.10) holds for any function f provided that $B = \{0\}$. Thus (cf. Step 2), (2.10) holds whenever the function f is continuous and strictly positive everywhere on \mathbb{R} and the set B is any Borel subset of $[0, \infty)$.

Step 4. Since the σ -algebra generated by the set of all bounded continuous strictly positive on \mathbb{R} functions is the entire Borel σ -algebra, we conclude by a

functional form of a monotone class argument that (2.10) holds whenever f is a nonnegative Borel function on \mathbb{R} (and the set B is any Borel subset of $[0, \infty)$.)

Step 5. Identity (2.10) (or its equivalent (2.12)) implies that (2.13) holds for all Borel functions F of the form $F(x, v) = \mathbf{I}\{x \in A, v \in B\}$. Then, again by a monotone class argument, (2.13) continues to hold for all nonnegative Borel functions F .

The proof of Theorem 2.1 is now complete. \square

Proof of Theorem 2.2. Take here the same Ω_0 as in the proof of Theorem 2.1. Then, by Lemma 3.9, on Ω_0 the r.v. Y is a strictly increasing (and hence one-to-one) transformation τ of r.v. W . Now Theorem 2.2 follows, with $\tilde{c} := c \circ \tau^{-1}$ and $\tilde{d} := d \circ \tau^{-1}$. \square

Proof of Theorem 2.3. The idea of the proof is simple. Since $X_1, \dots, X_n, U_1, \dots, U_n$ are all independent and, for each i , the r.v. W_i is a function of X_i and U_i , it follows that the pairs $(X_1, W_1), \dots, (X_n, W_n)$ are independent. Therefore, for each i , the conditional distribution of X_i given W_1, \dots, W_n is the same as that of X_i given W_i . By Theorem 2.1, the latter conditional distribution coincides a.s. with the unique zero-mean distribution on the set $\{c_i(W_i), d_i(W_i)\}$, where the functions c_i and d_i are constructed based on the (original, unconditional) distribution of X_i the way the functions c and d were constructed in the proof of part (i) of Theorem 2.1 based on the distribution of X ; at that, $d_i(W_i) - c_i(W_i) = W_i$ a.s. Hence, conditionally on W_1, \dots, W_n , the r.v.'s

$$\tilde{Z}_i := \frac{X_i}{\frac{1}{2}\sqrt{W_1^2 + \dots + W_n^2}}, \quad i = 1, \dots, n,$$

are independent and each \tilde{Z}_i is zero-mean and takes on (at most) two values,

$$\tilde{c}_i := \frac{c_i(W_i)}{\frac{1}{2}\sqrt{W_1^2 + \dots + W_n^2}} \quad \text{and} \quad \tilde{d}_i := \frac{d_i(W_i)}{\frac{1}{2}\sqrt{W_1^2 + \dots + W_n^2}},$$

so that $\tilde{d}_i - \tilde{c}_i = W_i / (\frac{1}{2}\sqrt{W_1^2 + \dots + W_n^2})$ a.s., whence a.s.

$$\frac{1}{2} \sqrt{\sum_{i=1}^n (\tilde{d}_i - \tilde{c}_i)^2} = 1.$$

This implies that, for all nonnegative Borel functions f

$$\mathbf{E}(f(S_W) | W_1, \dots, W_n) \leq \max(f(0), \sup \mathbf{E}f(Y_1 + \dots + Y_n))$$

a.s., where the sup is described in the statement of Theorem 2.3. Now inequality (2.16) follows.

Let us now give a formal proof of this inequality; it is based on Lemma 3.16.

Since $W_i \geq 0$ a.s. for all $i = 1, \dots, n$, integral $\int_{\mathbb{R}^n}$ in (3.1.24) can be replaced by $\int_{[0, \infty)^n}$. Therefore, under the conditions of Lemma 3.16, one has the inequality

$$(3.2.1) \quad \begin{aligned} \mathbf{E}F(X_1, W_1, \dots, X_n, W_n) \\ \leq \sup\{\mathbf{E}F(D_{v_1}^{(1)}, v_1, \dots, D_{v_n}^{(n)}, v_n) : (v_1, \dots, v_n) \in [0, \infty)^n\}. \end{aligned}$$

Now, for any nonnegative Borel function f on \mathbb{R} , let

$$F_f(x_1, v_1, \dots, x_n, v_n) := \begin{cases} f\left(\frac{x_1 + \dots + x_n}{\frac{1}{2}\sqrt{v_1^2 + \dots + v_n^2}}\right) & \text{if } v_1^2 + \dots + v_n^2 \neq 0, \\ f(0) & \text{otherwise.} \end{cases}$$

Note that, for $i = 1, \dots, n$, the r.v.'s

$$\tilde{Z}_i := \begin{cases} \frac{D_{v_n}^{(n)}}{\frac{1}{2}\sqrt{v_1^2 + \dots + v_n^2}} & \text{if } v_1^2 + \dots + v_n^2 \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

are independent, and each \tilde{Z}_i is zero-mean and – provided that $v_1^2 + \dots + v_n^2 \neq 0$ – takes on (at most) two values,

$$\tilde{c}_i := \frac{c_i(v_i)}{\frac{1}{2}\sqrt{v_1^2 + \dots + v_n^2}} \quad \text{and} \quad \tilde{d}_i := \frac{d_i(v_i)}{\frac{1}{2}\sqrt{v_1^2 + \dots + v_n^2}},$$

so that $\tilde{d}_i - \tilde{c}_i = v_i/(\frac{1}{2}\sqrt{v_1^2 + \dots + v_n^2})$ and

$$\frac{1}{2}\sqrt{\sum_{i=1}^n (\tilde{d}_i - \tilde{c}_i)^2} = 1.$$

This and inequality (3.2.1) imply inequality (2.16) for all nonnegative Borel functions f . \square

Proof of Theorem 2.5. This proof is quite similar to that of Theorem 2.3, using Theorem 2.2 in place of Theorem 2.1. \square

3.3. Proofs of the lemmata.

Proof of Lemma 3.1. For every $x \in \mathbb{R} \setminus (M \cup \{0\})$, let Δ_x denote the union of the set, say \mathcal{J}_x , of all (closed, open, or semi-open) intervals δ such that $\delta \ni x$ and $P(X \in \delta) = 0$. Then Δ_x is an interval. (Indeed, if x_1 and x_2 are in Δ_x , then $x_1 \in \delta_1 \subseteq \Delta_x$ and $x_2 \in \delta_2 \subseteq \Delta_x$ for some intervals $\delta_1 \in \mathcal{J}_x$ and $\delta_2 \in \mathcal{J}_x$; it follows that the union $\delta_1 \cup \delta_2$ is an interval which is an element of the set \mathcal{J}_x , and also $\delta_1 \cup \delta_2 \supseteq \{x_1, x_2\}$. Thus, for every two points x_1 and x_2 which are in Δ_x , all the points between x_1 and x_2 are also in Δ_x , so that Δ_x is an interval.) Moreover, the interval Δ_x is non-empty and, furthermore, it is of nonzero length, because, by the definition of M , for every $x \in \mathbb{R} \setminus (M \cup \{0\})$, the interval Δ_x contains an interval of the form $(y, x]$ for some $y < x$ or of the form $[x, y)$ for some $y > x$.

Observe next that, for every $x \in \mathbb{R} \setminus (M \cup \{0\})$, one has $P(X \in \Delta_x) = 0$. Indeed, assuming that $x \in \mathbb{R} \setminus (M \cup \{0\})$, let $[a, b]$ be any closed subinterval of Δ_x . Then there exist intervals δ_a and δ_b in \mathcal{J}_x such that $a \in \delta_a$ and $b \in \delta_b$. Hence, $x \in \delta_a \cap \delta_b$, $P(X \in \delta_a) = 0$, and $P(X \in \delta_b) = 0$, so that $[a, b] \subseteq \delta_a \cup \delta_b$, which implies $P(X \in [a, b]) \leq P(X \in \delta_a) + P(X \in \delta_b) = 0$. Thus, $P(X \in [a, b]) = 0$ for every closed subinterval $[a, b]$ of Δ_x . If the interval Δ_x is itself closed, this implies that $P(X \in \Delta_x) = 0$. If, for instance, Δ_x is a (necessarily non-empty) interval $[c, d)$, semi-open on the right, and $d_n \uparrow d$, then $P(X \in \Delta_x) = \lim_n P(X \in [c, d_n]) = 0$. The cases when the interval Δ_x is open or semi-open on the left are considered similarly. This proves the observation.

Observe further that, for any two points x and y in $\mathbb{R} \setminus (M \cup \{0\})$, the intervals Δ_x and Δ_y are either disjoint or the same. Indeed, suppose that (i) Δ_x and Δ_y are not disjoint and (ii) $\Delta_y \setminus \Delta_x \neq \emptyset$ (for instance). Then $\Delta := \Delta_x \cup \Delta_y \in \mathcal{J}_x$, while $\Delta \not\subseteq \Delta_x$; this contradicts the definition of Δ_x .

Therefore, the set $\{\Delta_x : x \in \mathbb{R} \setminus (M \cup \{0\})\}$ coincides (for some index set I) with a set $\{\delta_i : i \in I\}$ of intervals of nonzero length such that $\delta_i \cap \delta_j = \emptyset$ for any two different indices i and j in I . For every $i \in I$, one can choose a rational point $r_i \in \delta_i$, and these points will necessarily be distinct, since the intervals δ_i are disjoint. Therefore, the index set I must be countable. Since $x \in \Delta_x$ for every $x \in \mathbb{R} \setminus (M \cup \{0\})$, one concludes that

$$\begin{aligned} 0 \leq \mathbb{P}(X \notin M \cup \{0\}) &\leq \mathbb{P}\left(X \in \bigcup_{x \in \mathbb{R} \setminus (M \cup \{0\})} \Delta_x\right) \\ &= \mathbb{P}\left(X \in \bigcup_{i \in I} \delta_i\right) = \sum_{i \in I} \mathbb{P}(X \in \delta_i) = 0, \end{aligned}$$

because each δ_i coincides with some of the Δ_x 's. Now Lemma 3.1 follows. \square

Proof of Lemma 3.2. Let $h \in (0, m]$. Since $m = G(\infty) = \lim_{x \uparrow \infty} G(x)$, there exists some $x \in (0, \infty]$ such that $G(x) \geq h$. For any such x , (2.3) implies $x_+(h) \leq x$. Moreover, the right-continuity of G on $[0, \infty)$ implies $G(x_+(h)) \geq h$ (the latter inequality is trivial if $x_+(h) = \infty$). The inequality $G(x_+(h)) \geq h$, together with $h > 0$ and $G(0) = 0$, yields $x_+(h) \neq 0$. Thus, one has (3.1.6), which, in turn, implies (3.1.8). Relations (3.1.7) and (3.1.9) are verified similarly. The last sentence in Lemma 3.2 is now obvious. \square

Proof of Lemma 3.3. (I) Take any $h \in (0, m)$. At this point, let us check that $(x_+(h), u_+(h)) \in \mathcal{G}_+$. In other words, let us check that requirements (3.1.2) are satisfied if x and u are replaced there by $x_+(h)$ and $u_+(h)$, respectively.

(I)(i) Here we shall check that requirement (3.1.2a) is satisfied if x and u are replaced there by $x_+(h)$ and $u_+(h)$, respectively. That $0 \leq u_+(h) \leq 1$ follows immediately from (3.1.10) and the second part of (3.1.8).

It remains at this point to check that $x_+(h) \in M_+$. By Lemma 3.2, $x_+(h) \in (0, \infty)$. Assuming now that $x_+(h) \notin M_+$, one has $\mathbb{P}(X \in (y, x_+(h)]) = 0$ for some $y \in [0, x_+(h))$, so that $G(y) = G(x_+(h)) - \mathbb{E}X \mathbf{I}\{y < X \leq x_+(h)\} = G(x_+(h)) \geq h$, which contradicts the first part of (3.1.8). Thus, requirement (3.1.2a) is checked.

(I)(ii) It follows immediately from (3.1.10) that requirement (3.1.2b) is satisfied if x and u are replaced there by $x_+(h)$ and $u_+(h)$ respectively.

(I)(iii) Here we shall check condition (3.1.2c) for $x_+(h)$ and $u_+(h)$ in place of x and u . In view of point (I)(ii) above, one may assume that $x_+(h) \in L_+ \setminus N_+$ but $u_+(h) = 0$. Then $\exists y \in [0, x_+(h))$ $\mathbb{P}(X \in (y, x_+(h))) = 0$, and (3.1.10) implies that $G(x_+(h)-) = h$. Hence, $G(y) = G(x_+(h)-) - \mathbb{E}X \mathbf{I}\{X \in (y, x_+(h))\} = G(x_+(h)-) = h$, which contradicts the first part of (3.1.8).

(I)(iv) Let us now check condition (3.1.2d) for $x_+(h)$ and $u_+(h)$ in place of x and u . Assume that $\mathbb{P}(X > x_+(h)) = 0$. Then $G(x_+(h)) = m$. If $x_+(h) \in N_+$, then $G(x_+(h)-) = G(x_+(h)) = m > h$, which contradicts the second part of (3.1.8). Hence, $x_+(h) \notin N_+$. If now $u_+(h) = 1$, then (3.1.10) implies $G(x_+(h)) = h$, which is in a contradiction with $G(x_+(h)) = m > h$.

The verification of point (I) is now complete.

(II) Let us check next that map (3.1.11) is onto \mathcal{G}_+ . Take any $(x, u) \in \mathcal{G}_+$ and let

$$(3.3.1) \quad h := G(x-) + u \cdot (G(x) - G(x-)).$$

We need to check that (i) $h \in (0, m)$, (ii) $x_+(h) = x$, and (iii) $u_+(h) = u$.

(II)(i) Here we shall check that $h \in (0, m)$. Indeed, the condition $(x, u) \in \mathcal{G}_+$ implies $x \in M_+$, so that $P(X \in (0, x]) > 0$ and hence $G(x) > 0$. If $G(x-) > 0$, then (3.3.1) implies $h > 0$.

Consider now the case $G(x-) = 0$. Then $x \notin N_+$, because $G(x) > 0$. Also, here $x \in L_+$, because the equalities $G(x-) = 0 = G(0)$ imply $P(X \in (0, x)) = 0$. Therefore, conditions $(x, u) \in \mathcal{G}_+$ and (3.1.2c) imply that $u > 0$, so that (3.3.1) yields $h = uG(x) > 0$. Thus, $h > 0$ in all cases.

It remains at this point to check that $h < m$. This follows from (3.3.1) in the case $G(x) < m$, because $G(x-) \leq G(x)$ and $0 \leq u \leq 1$. Since $G(x) \leq G(\infty) = m$, it remains here to consider the case $G(x) = m$. Then one has $P(X > x) = 0$, so that, by (3.1.2d), $x \notin N_+$ and $u < 1$. Now (3.3.1) implies $h < G(x) = m$. Thus, $h < m$ in all cases.

(II)(ii) Here we shall check that $x_+(h) = x$. Take any $y \in [0, x)$. (Such a y exists since $x \in M_+ \subseteq (0, \infty)$.) To obtain a contradiction, suppose that $h \leq G(y)$. Then $h \leq G(x-)$. On the other hand, conditions (3.3.1) and $0 \leq u \leq 1$ imply $h \geq G(x-)$. Hence, $h = G(x-)$, and then (3.3.1) implies $u \cdot (G(x) - G(x-)) = 0$, which in turn implies that either $x \in N_+$ or $x \notin L_+$ (indeed, if $x \notin N_+$, then $u \cdot (G(x) - G(x-)) = 0$ implies $u = 0$, so that, by (3.1.2c), one has $x \notin L_+$). Taking now (3.1.1) into account, it follows now that $x \in M_+ \cap (N_+ \cup L_+^c) = L_+^c$, where we let $L_+^c := (0, \infty) \setminus L_+$, for brevity. Hence, for all $y \in [0, x)$ such that $h \leq G(y)$ one has $P(X \in (y, x)) > 0$, so that $h = G(x-) > G(y)$, a contradiction. Thus, $G(y) < h$ for all $y \in [0, x)$. On the other hand, (3.3.1) and $0 \leq u \leq 1$ imply $h \leq G(x)$. Now (3.1.6) yields $x_+(h) = x$.

(II)(iii) Here we shall check that $u_+(h) = u$. This follows from (3.1.10), (3.3.1), and (II)(ii) in the case $x \notin N_+$. If $x \in N_+$, then, by (3.1.2b), $u = 1$, so that $u_+(h) = u$ by (3.1.10).

The verification of point (II) is now complete.

(III) Let us check next that map (3.1.11) is one-to-one and its inverse is given by (3.3.1). Indeed, it follows by the first line of (3.1.10) in the case $x \notin N_+$ and by the second part of (3.1.8) in the case $x \in N_+$ that, if $x_+(h) = x$ and $u_+(h) = u$, then the value of h is given by (3.3.1), and is thus uniquely determined by x and u .

Thus, the first half of Lemma 3.3 is proved. The proof of its second half is quite similar. \square

Proof of Lemma 3.5. That x_+ , $-x_-$, and w are nonnegative and nondecreasing on $[0, m]$ and positive on $(0, m]$ follows immediately from (2.3), (2.4), Lemma 3.2, and (3.1.17).

Let now $h \in (0, m]$, $h_n \uparrow h$, $x_n := x_+(h_n)$, and $x := x_+(h)$. Then, because x_+ is nondecreasing, one has $x_n \nearrow y$ for some $y \in (0, x]$.

To obtain a contradiction, assume that $y < x$. Let $z \in (y, x)$. Then, by the first part of (3.1.8), $G(z) < h$. On the other hand, $y \geq x_n$ for all n . Hence, $h > G(z) \geq G(y) \geq G(x_n) \geq h_n$, by the second part of (3.1.8). This implies $h > G(z) \geq h$, which is a contradiction.

It follows that x_+ is left-continuous on $(0, m]$; similarly, for x_- and, in view of (3.1.17), for w . \square

Proof of Lemma 3.6. To obtain a contradiction, assume that $x_+(h_2) - x_+(h_1) < 0$. Then $h_2 < h_1$, since x_+ is nondecreasing (by Lemma 3.5). Hence, again by Lemma 3.5, $x_-(h_1) - x_-(h_2) \leq 0$. By re-grouping terms, it follows that

$$(3.3.2) \quad 0 \leq \varepsilon = w(h_2) - w(h_1) = (x_+(h_2) - x_+(h_1)) + (x_-(h_1) - x_-(h_2)) < 0,$$

a contradiction. Therefore, $x_+(h_2) - x_+(h_1) \geq 0$. Similarly is shown that $x_-(h_1) - x_-(h_2) \geq 0$. Now Lemma 3.6 follows from (3.3.2). \square

Proof of Lemma 3.9. In view of Remark 3.8, the function $\tau := |c|d$ satisfies (3.1.20) (in fact, this is the only such function). By Lemma 3.7, functions $|c|$ and d are nondecreasing and vanish only at 0, and also (in view of (3.1.18)) $|c|(v) + d(v) = v$ for all $v \in V \cup \{0\} \rightarrow \mathbb{R}$. It remains to show that τ is strictly increasing. Take any v_1 and v_2 in $V \cup \{0\} \rightarrow \mathbb{R}$ such that $0 \leq v_1 < v_2$. Then $\tau(v_2) = |c|(v_2)d(v_2) > 0$, $0 \leq |c|(v_1) \leq |c|(v_2)$, and $0 \leq d(v_1) \leq d(v_2)$. So, if $|c|(v_1) = 0$ or $d(v_1) = 0$, then $\tau(v_1) = 0 < \tau(v_2)$. Also, the identity $|c|(v) + d(v) = v$ implies that at least one of the inequalities $|c|(v_1) \leq |c|(v_2)$ and $d(v_1) \leq d(v_2)$ must be strict. Therefore, in all cases $\tau(v_1) = |c|(v_1)d(v_1) < |c|(v_2)d(v_2) = \tau(v_2)$. \square

Proof of Lemma 3.10. Let $v \in V$. Let us prove part (i) of Lemma 3.10. Accordingly, assume that $(x, u) \in \mathcal{G}_+$. In view of (3.1.22), (3.1.19), and (3.1.17), one has

$$(3.3.3) \quad \begin{aligned} d(v) &= d(w(h_v)) = x_+(h_v), & c(v) &= c(w(h_v)) = x_-(h_v), \\ v &= w(h_v) = x_+(h_v) - x_-(h_v). \end{aligned}$$

Let now (recall (3.1.12))

$$(3.3.4) \quad h := h_+(x, u),$$

so that, by Lemma 3.3 and definitions (2.5) and (3.1.17),

$$(3.3.5) \quad \begin{aligned} h &\in (0, m), \quad x = x_+(h) > 0, \quad r(x, u) = x_-(h) < 0, \\ |x - r(x, u)| &= x - r(x, u) = w(h). \end{aligned}$$

Now let us prove the “ \implies ” implication of part (i) of Lemma 3.10. Assume that $|x - r(x, u)| \leq v$, which can be rewritten, in view of the last equality in (3.3.5), as $w(h) \leq v$. Now it follows from (3.1.21) that

$$(3.3.6) \quad h \leq h_v.$$

Moreover, (3.3.5) and (3.3.3) together with Lemma 3.5 imply that $x \leq d(v)$. Thus, in view of (3.3.4) and (3.3.6), the “ \implies ” implication is checked.

Next, let us prove the “ \impliedby ” implication of part (i) of Lemma 3.10. Indeed, consider first the case $x < d(v)$, which can be rewritten, again in view of (3.3.5) and (3.3.3), as $x_+(h) < x_+(h_v)$; then, by the “nondecreasing” part of Lemma 3.5 and (3.1.22), one has $h < h_v$ and hence $|x - r(x, u)| = w(h) \leq w(h_v) = v$. Consider the remaining case when $x = d(v)$ & $h \leq h_v$. Then, applying (3.3.5), Lemma 3.5, and (3.1.22), one obtains $|x - r(x, u)| = w(h) \leq w(h_v) = v$. Thus, the “ \impliedby ” implication is also checked.

Thereby, part (i) of Lemma 3.10 is proved. Part (ii) of the lemma is proved similarly. \square

Proof of Lemma 3.11. Recalling the definitions of \mathcal{G} , \mathcal{G}_+ , and \mathcal{G}_- ((3.1.4), (3.1.2), (3.1.3)) and the relations $M_+ \subseteq (0, \infty)$ and $M_- \subseteq (-\infty, 0)$, one has

$$(3.3.7) \quad \mathbb{P}(X \neq 0, (X, U_X) \notin \mathcal{G}) = \mathbb{P}(X > 0, (X, U_X) \notin \mathcal{G}_+) + \mathbb{P}(X < 0, (X, U_X) \notin \mathcal{G}_-).$$

Next,

$$(3.3.8) \quad \begin{aligned} 0 &\leq \mathbb{P}(X > 0, (X, U_X) \notin \mathcal{G}_+) \\ &\leq \mathbb{P}(X > 0, X \notin M_+) + \mathbb{P}(U = 0) + \mathbb{P}(X \in K_+) + \mathbb{P}(U = 1), \end{aligned}$$

where

$$\begin{aligned} K_+ &:= \{x \in (0, \infty) : \mathbb{P}(X > x) = 0, x \in N_+\} \\ &= \{x \in (0, \infty) : \mathbb{P}(X \geq x) = 0\}. \end{aligned}$$

The four summands in (3.3.8) correspond to the restrictions on (x, u) in the definition of \mathcal{G}_+ . Namely, the first two summands correspond to restrictions (3.1.2a) and (3.1.2c), respectively, while the last two summands correspond to (3.1.2d). Note that restriction (3.1.2b) is already taken care of by definition (2.7) of U_x .

The second and the fourth summands in (3.3.8) are zero, because r.v. U is uniformly distributed between 0 and 1. The first summand is zero by Lemma 3.1. If $K_+ = \emptyset$, then the third summand is zero as well.

Assume now that $K_+ \neq \emptyset$. Observe that, if $x \in K_+$ and $y \in (x, \infty)$, then $0 \leq \mathbb{P}(X \geq y) \leq \mathbb{P}(X \geq x) = 0$, whence $y \in K_+$. This implies that K_+ is an interval, either of the form (a, ∞) for some $a \in [0, \infty)$ or of the form $[a, \infty)$ for some $a \in (0, \infty)$.

Therefore, if $a \in K_+$, then $K_+ = [a, \infty)$, and so, $\mathbb{P}(X \in K_+) = \mathbb{P}(X \geq a) = 0$. In the other case, when $a \notin K_+$, one has $K_+ = (a, \infty)$, and so, $\mathbb{P}(X \in K_+) = \mathbb{P}(X > a) = \lim_{n \rightarrow \infty} \mathbb{P}(X \geq a + \frac{1}{n}) = 0$. Thus, in all cases the third summand in (3.3.8) is zero.

Hence, $\mathbb{P}(X > 0, (X, U_X) \notin \mathcal{G}_+) = 0$. Similarly, $\mathbb{P}(X < 0, (X, U_X) \notin \mathcal{G}_-) = 0$. Now Lemma 3.11 follows by (3.3.7). \square

Proof of Lemma 3.12. Take any $v \in [0, \infty)$. In view of Lemma 3.11 and formulas (3.1.4) and (3.1.5),

$$\begin{aligned} &\mathbb{E} X \mathbf{I}\{|X - r(X, U_X)| \leq v\} \\ (3.3.9) \quad &= \mathbb{E} X \mathbf{I}\{(X, U_X) \in \mathcal{G}, |X - r(X, U_X)| \leq v\} \\ &= \mathbb{E} X \mathbf{I}\{(X, U_X) \in \mathcal{G}_+, |X - r(X, U_X)| \leq v\} \\ (3.3.10) \quad &+ \mathbb{E} X \mathbf{I}\{(X, U_X) \in \mathcal{G}_-, |X - r(X, U_X)| \leq v\} \end{aligned}$$

From this point on, the proof proceeds differently depending on properties of the value of v . We consider separately the following cases: (I) $V \cap (0, v] = \emptyset$; (II) $v \in V$; (III) v is any upper bound of V ; and (IV) $v \in (v_1, v_2)$ for some v_1 and v_2 in V . These cases are clearly exhaustive. However, in general, not all of these cases are mutually exclusive.

(I) Consider first the case $V \cap (0, v] = \emptyset$. By Lemma 3.3 and (3.1.18), $(x, u) \in \mathcal{G}$ implies that $|x - r(x, u)| \in V$. Therefore, the expression in (3.3.9) is zero. Thus, Lemma 3.12 is proved in the case $V \cap (0, v] = \emptyset$.

(II) Next, consider the case $v \in V$.

In this case, by Lemma 3.10 and also again Lemma 3.11,

$$\begin{aligned}
& \mathbb{E}X \mathbf{I}\{(X, U_X) \in \mathcal{G}_+, |X - r(X, U_X)| \leq v\} \\
&= \mathbb{E}X \mathbf{I}\{(X, U_X) \in \mathcal{G}_+, (X < d(v) \text{ or } (X = d(v) \ \& \ h_+(X, U_X) \leq h_v))\} \\
&= \mathbb{E}X \mathbf{I}\{0 < X < d(v) \text{ or } (X = d(v) \ \& \ h_+(d(v), U_{d(v)}) \leq h_v)\} \\
(3.3.11) \quad &= G(d(v)-) + d(v)\mathbb{P}(X = d(v))\mathbb{P}(h_+(d(v), U) \leq h_v);
\end{aligned}$$

the last equality is obvious if $d(v) \in N_+$, and it follows from the definition (2.7) and the independence of X and U if $d(v) \notin N_+$.

Note that

$$d(v)\mathbb{P}(X = d(v)) = G(d(v)) - G(d(v)-).$$

Recall that, in view of (3.1.22) and (3.1.19),

$$(3.3.12) \quad d(v) = d(w(h_v)) = x_+(h_v).$$

If $h_v = m$ then, by Lemma 3.5 and (3.1.22), $w(h) \leq w(h_v) = v$ for all $h \in [0, m]$; that is, v is an upper bound of the set V , so that one has Case (III), to be considered next.

It remains here to consider the case $h_v < m$.

Consider the two possible subcases.

Subcase 1: $d(v) \in N_+$. In view of (3.3.12) and (3.1.12), for any $u \in [0, 1]$, the expression in (3.3.11) equals

$$G(d(v)-) = G(d(v)-) + u \cdot (G(d(v)) - G(d(v)-)) = h_+(d(v), u) = h_+(x_+(h_v), u).$$

Now, substituting here $u_+(h_v)$ for u , one has by Lemma 3.3 that, in Subcase 1, the expression in (3.3.11) equals h_v .

Subcase 2: $d(v) \notin N_+$. Here, in view of (3.3.12), (3.1.12), and (3.1.10), one has

$$\begin{aligned}
\mathbb{P}(h_+(d(v), U) \leq h_v) &= \mathbb{P}(h_+(x_+(h_v), U) \leq h_v) \\
&= \mathbb{P}(G(x_+(h_v)-) + U \cdot (G(x_+(h_v)) - G(x_+(h_v)-)) \leq h_v) \\
&= \mathbb{P}(U \leq u_+(h_v)) = u_+(h_v).
\end{aligned}$$

Hence, in Subcase 2, the expression in (3.3.11) equals

$$\begin{aligned}
G(d(v)-) + u_+(h_v) \cdot (G(d(v)) - G(d(v)-)) &= h_+(d(v), u_+(h_v)) \\
&= h_+(x_+(h_v), u_+(h_v)) = h_v,
\end{aligned}$$

by (3.1.12), (3.3.12), and Lemma 3.3.

Thus, in both Subcase 1 and Subcase 2, the expression in (3.3.11) equals h_v . That is, the first summand in (3.3.10) equals h_v . Similarly, the second summand in (3.3.10) equals $-h_v$. Now Lemma 3.12 follows – for all $v \in V$.

(III) Next, if v is any upper bound of V then, by (3.1.18) and Lemma 3.3, $(x, u) \in \mathcal{G}_+$ implies $|x - r(x, u)| \leq v$, so that, in view of Lemma 3.11, the first summand in (3.3.10) equals $\mathbb{E}X \mathbf{I}\{X > 0\} = m$; similarly, the second summand in (3.3.10) equals $-m$. Thus, Lemma 3.12 is proved in the case when v is any upper bound of V .

(IV) It remains to consider the case when $v \in (v_1, v_2)$ for some v_1 and v_2 in V , so that $v_i = w(h_i)$ for some $h_i \in (0, m]$, where $i = 1, 2$. Let

$$v_* := \sup(V \cap (0, v]).$$

Then $v_* \in (0, v]$ (because $v_1 \in V \cap (0, v]$ and hence $V \cap (0, v] \neq \emptyset$).

Moreover, $v_* \in V$, so that

$$(3.3.13) \quad v_* := \max(V \cap (0, v]).$$

Indeed, otherwise there is a strictly increasing sequence (v_n) in $V \cap (0, v]$ which converges to v_* . Then, by (3.1.18), there exists a sequence (h_n^*) in $(0, m]$ such that $v_n = w(h_n^*)$ for all n . By Lemma 3.5, the function w is nondecreasing, and so, the sequence (h_n^*) is necessarily increasing. Hence, $h := \lim_n h_n^* \in (0, m]$. Again by Lemma 3.5, the function w is left-continuous on $(0, m]$, and so, $w(h) = \lim_n w(h_n^*) = \lim_n v_n = v_*$. Thus, the claim that $v_* \in V$ is checked.

In view of Lemma 3.3 and (3.1.18), $(x, u) \in \mathcal{G}$ implies that $|x - r(x, u)| \in V$, whence, by (3.3.13), for all $(x, u) \in \mathcal{G}$,

$$|x - r(x, u)| \leq v \iff |x - r(x, u)| \leq v_*.$$

Therefore and by virtue of (3.3.9), the case when $v \in (v_1, v_2)$ for some v_1 and v_2 in V is reduced to case (II) $v \in V$. \square

Proof of Lemma 3.13. Lemma 3.12 implies

$$\mathbf{E}X \mathbf{I}\{W \in (v_1, v_2]\} = \mathbf{E}X \mathbf{I}\{W \leq v_2\} - \mathbf{E}X \mathbf{I}\{W \leq v_1\} = 0$$

for any left-open interval $(v_1, v_2] \subseteq (0, \infty)$. Thus, the countably additive function (c.a.f.) $B \mapsto \mathbf{E}X \mathbf{I}\{W \in B\}$ is zero on the semiring of such intervals. Since this semiring generates the entire Borel σ -algebra in $(0, \infty)$, this c.a.f. is zero on this σ -algebra. It remains to note that $\mathbf{E}X \mathbf{I}\{W \in \{0\}\} = \mathbf{E}X \mathbf{I}\{W \leq 0\} = 0$, because $W \geq 0$ a.s. and by Lemma 3.12. \square

Proof of Lemma 3.14. For any set $A \subseteq \mathbb{R}$, consider its pre-images under c and d :

$$d^{-1}(A) := \{v \in V : d(v) \in A\} \quad \text{and} \quad c^{-1}(A) := \{v \in V : c(v) \in A\}.$$

Then, for any $\delta \in (0, \varepsilon]$, the sets

$$C_{j,k} := d^{-1}((e^{\delta k}, e^{\delta(k+1)})) \cap c^{-1}([-e^{\delta(j+1)}, -e^{\delta j}]) \cap B,$$

where j and k run over all integers, form a partition of B which is both (d, ε) -good and (c, ε) -good (because, by Lemma 3.7, functions d and $-c$ are (strictly) positive on V).

It suffices to prove that this partition is also (f, ε) -good, provided that $\delta \in (0, \varepsilon]$ is small enough. Toward that end, consider any one of the $C_{j,k}$'s which are not null, so that

$$(3.3.14) \quad 0 < d_{\max}(C_{j,k}) \leq e^{\delta} d_{\min}(C_{j,k}),$$

by the construction of $C_{j,k}$.

Let

$$(3.3.15) \quad \delta_1 := (e^{\varepsilon} - 1) \inf\{f(x) : |x| \leq \sup B\}.$$

Then $\delta_1 > 0$, because the set B is assumed to be bounded and the function f , everywhere continuous and strictly positive. Then f is uniformly continuous on all bounded sets, so that there exists some $\delta_2 > 0$ such that

$$(|x - y| \leq \delta_2 \text{ \& } |x| \leq \sup B) \implies |f(x) - f(y)| \leq \delta_1.$$

Choose now $\delta \in (0, \varepsilon]$ to be small enough so that

$$(e^{\delta} - 1) \sup B \leq \delta_2.$$

Note that

$$0 \leq d_{\min}(C_{j,k}) \leq d_{\max}(C_{j,k}) \leq d_{\max}(B) \leq \sup B,$$

by Lemma 3.7. Therefore and in view of (3.3.14),

$$0 \leq d_{\max}(C_{j,k}) - d_{\min}(C_{j,k}) \leq (e^\delta - 1)d_{\min}(C_{j,k}) \leq (e^\delta - 1)\sup B \leq \delta_2,$$

and so,

$$0 < f_{r,\max}(C_{j,k}) \leq f_{r,\min}(C_{j,k}) + \delta_1 \leq e^\varepsilon f_{r,\min}(C_{j,k}),$$

by the definition (3.3.15) of δ_1 . Similarly,

$$0 < f_{\ell,\max}(C_{j,k}) \leq e^\varepsilon f_{\ell,\min}(C_{j,k}).$$

□

Proof of Lemma 3.15. The case when C is a null set is trivial, because then each of the three terms in (3.1.23) is zero.

Assume now that the set C is ε -good. If $(x, u) \in \mathcal{G}_+$, then, by Lemma 3.3, $x = x_+(h)$ for some $h \in (0, m)$; hence, by (3.1.19), $x = d(v)$ for $v := w(h) = x - r(x, u)$. Therefore, if event $\{(X, U_X) \in \mathcal{G}_+, W \in C\}$ occurs, then $X = d(W)$, whence $X \in [d_{\min}(C), d_{\max}(C)]$. Similarly, if event $\{(X, U_X) \in \mathcal{G}_-, W \in C\}$ occurs, then $X = c(W)$, whence $X \in [c_{\min}(C), c_{\max}(C)]$. Also, if event $\{W \in C\}$ occurs, then $X \neq 0$, because $C \subset (0, \infty)$, and $X = 0$ implies $W = 0$.

It follows that

$$(3.3.16) \quad \{W \in C\} \subseteq \{X \in [d_{\min}, d_{\max}] \cup [c_{\min}, c_{\max}]\} \cup E,$$

where

$$E := \{X \neq 0, (X, U_X) \notin \mathcal{G}\},$$

and we set, for brevity:

$$d_{\min} := d_{\min}(C), \quad d_{\max} := d_{\max}(C), \quad c_{\min} := c_{\min}(C), \quad c_{\max} := c_{\max}(C).$$

Note that, by Lemma 3.11,

$$(3.3.17) \quad \mathbf{P}(E) = 0.$$

It follows from Lemma 3.13, (3.3.16), and (3.3.17) that

$$(3.3.18) \quad 0 = \mathbf{E}X \mathbf{I}\{W \in C\} \leq c_{\max}q + d_{\max}p,$$

where

$$p := \mathbf{P}(X \in [d_{\min}, d_{\max}], W \in C), \quad q := \mathbf{P}(X \in [c_{\min}, c_{\max}], W \in C),$$

so that

$$(3.3.19) \quad p + q = \mathbf{P}(W \in C).$$

Similarly,

$$(3.3.20) \quad 0 = \mathbf{E}X \mathbf{I}\{W \in C\} \geq c_{\min}q + d_{\min}p,$$

It follows from (3.3.18) and (3.3.20) that

$$(3.3.21) \quad \frac{-c_{\max}}{d_{\max}}q \leq p \leq \frac{-c_{\min}}{d_{\min}}q.$$

Next, letting

$$f_{r,\min} := f_{r,\min}(C), \quad f_{r,\max} := f_{r,\max}(C), \quad f_{\ell,\min} := f_{\ell,\min}(C), \quad f_{\ell,\max} := f_{\ell,\max}(C),$$

one has

$$(3.3.22) \quad \mathbb{E}f(X) \mathbf{I}\{W \in C\} \leq f_{\ell, \max} q + f_{r, \max} p$$

$$(3.3.23) \quad \leq \frac{q}{d_{\min}} (f_{\ell, \max} \cdot d_{\min} + f_{r, \max} \cdot (-c_{\min}));$$

here, inequality (3.3.22) is similar to (3.3.18), and (3.3.23) follows from the second inequality in (3.3.21).

On the other hand, recalling (2.11) and (3.3.19), and then also using the first inequality in (3.3.21), one obtains

$$(3.3.24) \quad \begin{aligned} \mathbb{E}\varphi_f(W) \mathbf{I}\{W \in C\} &\geq \frac{f_{\ell, \min} \cdot d_{\min} + f_{r, \min} \cdot (-c_{\max})}{-c_{\min} + d_{\max}} (p + q) \\ &\geq \frac{f_{\ell, \min} \cdot d_{\min} + f_{r, \min} \cdot (-c_{\max})}{-c_{\min} + d_{\max}} \frac{q}{d_{\max}} (-c_{\max} + d_{\max}) \\ &= r_1 r_2 r_3 \frac{q}{d_{\min}} (f_{\ell, \max} \cdot d_{\min} + f_{r, \max} \cdot (-c_{\min})), \end{aligned}$$

where

$$\begin{aligned} r_1 &:= \frac{d_{\min}}{d_{\max}} \geq e^{-\varepsilon}, \\ r_2 &:= \frac{-c_{\max} + d_{\max}}{-c_{\min} + d_{\max}} \geq e^{-\varepsilon}, \\ r_3 &:= \frac{f_{\ell, \min} \cdot d_{\min} + f_{r, \min} \cdot (-c_{\max})}{f_{\ell, \max} \cdot d_{\min} + f_{r, \max} \cdot (-c_{\min})} \geq e^{-\varepsilon} \frac{d_{\min} + (-c_{\max})}{d_{\min} + (-c_{\min})} \geq e^{-2\varepsilon}. \end{aligned}$$

Now it follows from (3.3.24) and (3.3.23) that

$$\mathbb{E}\varphi_f(W) \mathbf{I}\{W \in C\} \geq e^{-4\varepsilon} \mathbb{E}f(X) \mathbf{I}\{W \in C\},$$

which proves the second inequality in (3.1.23). The first inequality in (3.1.23) is proved quite similarly. \square

Proof of Lemma 3.16. For $j = 1, \dots, n+1$, introduce

$$(3.3.25) \quad \begin{aligned} &\Phi_j(x_1, v_1, \dots, x_{j-1}, v_{j-1}; v_j, v_{j+1}, \dots, v_n) \\ &:= \mathbb{E}F(x_1, v_1, \dots, x_{j-1}, v_{j-1}, D_{v_j}^{(j)}, v_j, \dots, D_{v_n}^{(n)}, v_n) \end{aligned}$$

and

$$(3.3.26) \quad \begin{aligned} \mathcal{I}_j &:= \int_{\mathbb{R}^{n-1+j}} \left(\prod_{i=1}^{j-1} \mathbb{P}(X_i \in dx_i, W_i \in dv_i) \right) \left(\prod_{i=j}^n \mathbb{P}(W_i \in dv_i) \right) \\ &\quad \times \Phi_j(x_1, v_1, \dots, x_{j-1}, v_{j-1}; v_j, v_{j+1}, \dots, v_n). \end{aligned}$$

Then, for all $j = 1, \dots, n$,

$$(3.3.27) \quad \mathcal{I}_{j+1} = \int_{\mathbb{R}^{n-2+j}} \left(\prod_{i=1}^{j-1} \mathbb{P}(X_i \in dx_i, W_i \in dv_i) \right) \left(\prod_{i=j+1}^n \mathbb{P}(W_i \in dv_i) \right) \mathcal{E}_j,$$

where

$$\begin{aligned}
\mathcal{E}_j &:= \mathbf{E}\Phi_{j+1}(x_1, v_1, \dots, x_{j-1}, v_{j-1}, X_j, W_j; v_{j+1}, \dots, v_n) \\
&= \int_{\mathbb{R}} \mathbf{P}(W_j \in dv_j) \mathbf{E}\Phi_{j+1}(x_1, v_1, \dots, x_{j-1}, v_{j-1}, D_{v_j}^{(j)}, v_j; v_{j+1}, \dots, v_n) \\
&= \int_{\mathbb{R}} \mathbf{P}(W_j \in dv_j) \int_{\mathbb{R}} \mathbf{P}(D_{v_j}^{(j)} \in dx_j) \\
&\quad \times \Phi_{j+1}(x_1, v_1, \dots, x_{j-1}, v_{j-1}, x_j, v_j; v_{j+1}, \dots, v_n) \\
&= \int_{\mathbb{R}} \mathbf{P}(W_j \in dv_j) \int_{\mathbb{R}} \mathbf{P}(D_{v_j}^{(j)} \in dx_j) \\
&\quad \times \mathbf{E}F(x_1, v_1, \dots, x_j, v_j, D_{v_{j+1}}^{(j+1)}, v_{j+1}, \dots, D_{v_n}^{(n)}, v_n) \\
&= \int_{\mathbb{R}} \mathbf{P}(W_j \in dv_j) \mathbf{E}F(x_1, v_1, \dots, x_{j-1}, v_{j-1}, D_{v_j}^{(j)}, v_j, \dots, D_{v_n}^{(n)}, v_n) \\
&= \int_{\mathbb{R}} \mathbf{P}(W_j \in dv_j) \Phi_j(x_1, v_1, \dots, x_{j-1}, v_{j-1}; v_j, \dots, v_n);
\end{aligned}$$

the second of these 6 equalities follows by (2.13), and the fourth and sixth ones by (3.3.25).

Now (3.3.27) and (3.3.26) imply that

$$\begin{aligned}
\mathcal{I}_{j+1} &= \int_{\mathbb{R}^{n-1+j}} \left(\prod_{i=1}^{j-1} \mathbf{P}(X_i \in dx_i, W_i \in dv_i) \right) \left(\prod_{i=j}^n \mathbf{P}(W_i \in dv_i) \right) \\
&\quad \times \Phi_j(x_1, v_1, \dots, x_{j-1}, v_{j-1}; v_j, v_{j+1}, \dots, v_n) \\
&= \mathcal{I}_j,
\end{aligned}$$

for all $j = 1, \dots, n$. This finally implies $\mathcal{I}_{n+1} = \mathcal{I}_1$, so that

$$\begin{aligned}
&\mathbf{E}F(X_1, W_1, \dots, X_n, W_n) = \mathcal{I}_{n+1} \\
&= \mathcal{I}_1 = \int_{\mathbb{R}^n} \left(\prod_{i=1}^n \mathbf{P}(W_i \in dv_i) \right) \mathbf{E}F(D_{v_1}^{(1)}, v_1, \dots, D_{v_n}^{(n)}, v_n).
\end{aligned}$$

□

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